On a generalisation of self-coupled conformally covariant spin-2 equations

1983 J. Phys. A: Math. Gen. 16 L37

(http://iopscience.iop.org/0305-4470/16/1/008)

J. Phys. A: Math. Gen. 16 (1983) L37–L40. Printed in Great Britain

LETTER TO THE EDITOR

On a generalisation of self-coupled conformally covariant spin-2 equations

Mark S Drew

Department of Computing Science, Simon Fraser University, Burnaby, BC, Canada V5A 1S6

Received 2 November 1982

Abstract. The symmetric tensor equation of Drew and Gegenberg was recently reinterpreted by Barut and Xu as the usual Fierz–Pauli field coupled to itself and to a scalar field. Here, this equation is generalised such that the scalar interaction has a non-unique coupling constant λ; this coupling constant affects both the field equations and the traceless version of the energy–momentum tensor with resulting effects on possible vacuum solutions.

Recently, Barut and Xu (1982) put forward an interesting interpretation of the alternative equation for a symmetric tensor field h_{ab} postulated by Drew and Gegenberg (1980). This equation is

$$\Box h_{ab} - \tfrac{2}{3}\partial_{(a}\partial^e h_{b)e} + \tfrac{1}{3}\partial_a\partial_b h + \tfrac{1}{3}\eta_{ab}\partial_p\partial_q h^{pq} = 0, \tag{1}$$

where

$$h_{ab} = h_{ba}, \qquad h \equiv h^a_a, \qquad \eta_{ab} = \mathrm{diag}(-1, -1, -1, 1), \qquad \partial_{(a}h_{b)e} = \partial_a h_{be} + \partial_b h_{ae}.$$

The above equation has the feature that it is conformally covariant, in the sense of Mack and Salam (1969); the usual linearised Einstein equation (the massless free Fierz–Pauli equation),

$$\Box h_{ab} - \partial_{(a}\partial^e h_{b)e} + \partial_a\partial_b h = 0, \tag{2}$$

is not conformally covariant.

By interpreting the *difference* between the new equation and the usual massless spin-2 equation (2) as a source term to be placed on the right-hand side of the Fierz–Pauli equation (2), Barut and Xu suggest that the equation of Drew and Gegenberg be interpreted as the symmetric tensor field coupled to itself and to a scalar field given by its trace. In this paper, it is pointed out that a generalisation of equation (1) leads to a non-unique coupling constant with the scalar field. This generalises the scalar interaction in Barut and Xu (1982).

The field equation postulated in Drew and Gegenberg (1980) was determined by requiring that the equation for h_{ab} could be factored via the spin-representation matrices s_{jk} in the same way as it was shown to be possible for the electromagnetic field, namely

$$(-\mathrm{i}\, s^{pq}\partial_q + n\,\partial^p)(-\mathrm{i}\, s_{pr}\partial^s + n\,\partial_p)\Phi = 0, \tag{3}$$

with n a constant and Φ either massless field. For a symmetric tensor field h_{ab} this equation yields equation (1).

Now, Barut and Xu make use of the traceless version of (1),

$$\Box h_{ab}-\tfrac{2}{3}\partial_{(a}\partial^{e}h_{b)e}+\tfrac{1}{3}\partial_a\partial_b h+\tfrac{1}{3}\eta_{ab}(\partial_p\partial_q h^{pq}-\Box h)=0. \tag{4}$$

This equation arises from a Lagrangian density $\boldsymbol{L}$ which can be put in the simple form

$$\boldsymbol{L}=\tfrac{1}{3}(\pi^{pq})^{r}(\pi_{pq})_{r}, \tag{5}$$

with $(\pi^{pq})^r$ the canonical momentum density with respect to $\partial_r h_{pq}$. Here, $(\pi^{pq})^r$ can be given in terms of the combination

$$G_{jkn}=h_{jk,n}-h_{jn,k}\equiv-G_{jnk}, \tag{6}$$
$$G_{j\ n}^{j}\equiv G_n, \qquad h_{jk,n}\equiv\partial_n h_{jk},$$

as

$$(\pi^{pq})^{r}=\tfrac{1}{2}G^{(pq)r}-(1/k)\eta^{pq}G^{r}+(1/2k)\eta^{r(p}G^{q)}, \tag{7}$$

with $k=3$. If one were to take $k=1$ instead, the usual equations (2) result. In the above formalism, the divergencelessness condition imposed on (4) by Barut and Xu amounts to

$$G^{n}{}_{,n}=0. \tag{8}$$

The trace part of (1) simply implies

$$\Box h=0, \tag{9}$$

which is easily shown to be conformally covariant if h is the trace of a symmetric tensor field; equation (9) is also the correct conformally covariant equation for a massless scalar free field h. Therefore, if the difference between (4) and (2) is to be interpreted as the conformally covariant interaction term between h_{ab} and itself and the scalar part h, then one should also include an additional multiple of $\Box h$, and write the full equation as

$$\Box h_{ab}-\tfrac{2}{3}\partial_{(a}\partial^{e}h_{b)e}+\tfrac{1}{3}\partial_a\partial_b h+\tfrac{1}{3}\eta_{ab}\partial_p\partial_q h^{pq}+\lambda\eta_{ab}\Box h=0. \tag{10}$$

Here, an additional coupling constant λ has been introduced, generalising the interaction given in Barut and Xu (1982); the restricted equation (4) in Barut and Xu results from $\lambda\equiv-\frac{1}{3}$. The presence of a coupling constant may have important bearing on the possibility of finding explicit solutions of the field equations (cf Barut and Xu 1981a, b).

To this end it would be useful to have an expression for a traceless energy–momentum tensor which could be set to zero in order to limit possible vacuum solutions. It is convenient to write a Lagrangian $\boldsymbol{L}$ as

$$\boldsymbol{L}\equiv\tfrac{1}{2}(\pi_{jk})_n h^{jk,n}\equiv\tfrac{1}{2}\pi_n\psi^{,n}, \tag{1}$$

with the coupling constant λ incorporated via

$$(\pi_{jk})_n\equiv P^{abc}_{jkn}h_{ab,c}=\partial\boldsymbol{L}/\partial(\partial^n h^{jk}), \tag{12}$$

with

$$\begin{aligned}P^{abc}_{jkn}&=P^{bac}_{jkn}=P^{abc}_{kjn}\\&=\tfrac{1}{6}\delta_n{}^{(a}\delta_{(j}{}^{b)}\delta_{k)}{}^{c}+\tfrac{1}{6}\eta_{n(j}\delta_{k)}{}^{(a}\eta^{b)c}-\tfrac{1}{6}\eta_{n(j}\eta^{ab}\delta_{k)}{}^{c}\\&\quad-\tfrac{1}{4}\delta_{(j}{}^{(a}\delta_{k)}{}^{b)}\delta_n{}^{c}-\tfrac{1}{6}\eta_{jk}\delta_n{}^{(a}\eta^{b)c}-\lambda\eta_{jk}\eta^{ab}\delta_n{}^{c}.\end{aligned} \tag{13}$$

In (11) and below, whenever contraction of $(\pi_{jk})_n$ and h^{jk} occurs, the product is written as $\pi_n\psi$.

Written our explicitly, one has

$$\boldsymbol{L} = -\tfrac{1}{2}h_{jk,n}h^{jk,n} - \tfrac{1}{3}h_{,k}h_n{}^{k,n} + \tfrac{1}{3}h_{nj,k}h^{jk,n} + \tfrac{1}{3}h_k{}^j{}_{,j}h_n{}^{k,n} - \tfrac{1}{2}h_{,n}h^{,n}, \tag{14}$$

$$(\pi_{jk})_n = \tfrac{1}{3}h_{n(j,k)} + \tfrac{1}{3}h_{(k}{}^c{}_{,c}\eta_{nj)} - \tfrac{1}{6}\eta_{n(j}h_{,k)} - h_{jk,n} - \tfrac{1}{3}h_n{}^c{}_{,c}\eta_{jk} - \lambda\eta_{jk}h_{,n}. \tag{15}$$

As the first step in finding a traceless energy–momentum tensor θ^{jk}, one must consider the product $\pi_n\psi$ (Callan *et al* 1970). For a free scalar field ϕ,

$$\boldsymbol{L} = \tfrac{1}{2}\pi_n\partial^n\phi, \qquad \pi_n = \partial_n\phi, \tag{16}$$

so that

$$\pi_n\phi = \tfrac{1}{2}\partial_n(\phi^2). \tag{17}$$

Similarly, for (13) one has

$$P^{abc}_{jkn}h^{jk}{}_{,c} = (\pi^{ab})_n, \tag{18}$$

so that

$$\tfrac{1}{2}\partial_c(P^{abc}_{jkn}h_{ab}h^{jk}) \equiv \pi_n\psi. \tag{19}$$

Therefore, the most straightforward way to find a (non-symmetric) conserved traceless θ_{jk} is analogy with the scalar case, namely

$$\theta_j{}^k = t_j{}^k + X_j{}^k, \tag{20}$$

where $t_j{}^k$ is the canonical energy–momentum tensor

$$t_j{}^k = \pi^k\partial_j\psi - \partial_j{}^k\boldsymbol{L} \tag{21}$$

and

$$X_j{}^k \equiv -\tfrac{1}{6}(P^{abk}_{cde}\partial_j\partial^e - P^{abe}_{cdf}\delta_j{}^k\partial_e\partial^f)(h_{ab}h^{cd}). \tag{22}$$

Here, it is easy to show explicitly that t_{jk} is conserved by virtue of the field equations $\pi^k{}_{,k} = 0$ by using the first of the relations

$$P^{abc}_{jkn}h^{jk,n} = (\pi^{ab})^c, \qquad P^{abc}_{jkn}h_{ab}{}^{,n} = (\pi_{jk})^c. \tag{23}$$

Now, X_{jk} is identically conserved,

$$X_j{}^k{}_{,k} \equiv 0, \tag{24}$$

and its trace is

$$X_j{}^j = \partial_j(\pi^j\psi), \tag{25}$$

while the trace of t_{jk} is

$$t_j{}^j = -\pi^j\partial_j\psi. \tag{26}$$

Therefore θ_{jk} is traceless if the field equations hold for $\psi \equiv h_{jk}$, and (20) does indeed give a traceless, conserved energy–momentum tensor.

To illustrate how the presence of λ might affect the existence of possible solutions, consider the simplest case $h_{ab} = \eta_{ab}f(x)$. In that case, the field equations (10) read

$$\tfrac{4}{3}(1+3\lambda)\eta_{ab}\Box f = 0 \tag{27}$$

(and recall that Barut and Xu have $\lambda \equiv -\frac{1}{3}$). The traceless energy–momentum tensor reads in this case

$$\theta^c{}_d = \tfrac{16}{9}(1+3\lambda)[\delta^c{}_d\tfrac{1}{2}(\partial_n f \partial^n f) - 2\partial_d f \partial^c f + f(\partial_d \partial^c f - \delta_d{}^c \Box f)]. \tag{28}$$

Therefore, it can be seen that without the extra coupling constant λ it is not possible in this case to find solutions of the field equations in conjunction with the conditions $\theta^c{}_d = 0$.

References

Barut A O and Xu B W 1981a *Phys. Lett.* **102B** 37
—— 1981b *Phys. Rev.* D **23** 3076
—— 1982 *J. Phys. A: Math. Gen.* **15** L207
Callan C G, Coleman S and Jackiw R 1970 *Ann. Phys., NY* **59** 42
Drew M S and Gegenberg J D 1980 *Nuovo Cimento* **60A** 41
Mack G and Salam A 1969 *Ann. Phys., NY* **53** 174